third place. It was also found that to fifty-two places of decimals neither $e^{\pi} \cdot e^{-\pi}$ nor $e^{\pi/2} \cdot e^{-(\pi/2)}$ differed from unity.

In this MONTHLY, *1917*, 237, it is stated that—for the value of $e^{-(\pi/2)}$ or $i^i$—"Mr. Escott using Steinhauser's 20-place tables, gets .20787957635076190854687 while Professor Reynolds, using Hutton's 20-place tables, gets .2078795763-4917907781." Comparison of these numbers with my datum given above shows that Mr. Escott's differs from it by only about one unit in the twenty-second decimal place, whereas Professor Reynolds's number is discordant at, and beyond, the eleventh place. Since this may mean that there is an error in Hutton's tables it would be helpful if Professor Reynolds would investigate and report the cause of the discrepancy.

Finally, in the seventh revised edition of the *Smithsonian Physical Tables*, 1920, page 55, a short table of values[1] of $e^{(\pi/4)x}$, $e^{-(\pi/4)x}$ and their logarithms is given. If it should ever become necessary to compute tables of this kind to a greater number of decimal places than can be effected advantageously by logarithms, my data given above may be used either as basic numbers or as independent checks.

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## HISTORICAL NOTES ON THE RELATION $e^{-(\pi/2)} = i^i$.

BY R. C. ARCHIBALD, Brown University.

In 1719 Count Giulio Carlo de'Toschi di Fagnano showed, in effect, that the arc of the quadrant of a unit circle ($\pi/2$) is[2]

$$2 \log.(1 - \sqrt{-1})^{\frac{1}{2}\sqrt{-1}} \times (1 + \sqrt{-1})^{-\frac{1}{2}\sqrt{-1}}.$$

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[1] For $x = 1, 2, 3, \cdots 20$. The part of the table without logarithms is also given on page 91 of J. B. Dale, *Five Figure Tables*. London, 1903. The values of $e^{19\pi/4}$ differ materially in these two sources.—EDITOR.

[2] *Opere Matematiche del Marchese . . . de'Toschi di Fagnano.* Milano, volume 2, 1912. On page 406 we find "$\int \frac{dt}{1+t^2}$ esprime l'arco di cerchio [radius unity], la di cui tangente é $t$" [if the equations of the circle are $x = \cos\theta$ and $y = \sin\theta$, $t = \tan\theta$]. On pages 422–423 we find the following:

$$\int \frac{dt}{1+t^2} = \log.\ (1 - t\sqrt{-1})^{\frac{1}{2}\sqrt{-1}} \times (1 + t\sqrt{-1})^{-\frac{1}{2}\sqrt{-1}}$$

or

$$\text{"(8)} \qquad \int \frac{dt}{1+t^2} = \log.\ (A^2 - B^2).$$

[where $A = 1 + \frac{1}{2}t + \frac{1}{8}t^2 - \frac{7}{48}t^3 - \frac{43}{384}t^4 \cdots$, $\quad B = \frac{1}{4}t^2\sqrt{-1} + \frac{1}{8}t^3\sqrt{-1} - \frac{3}{32}t^4\sqrt{-1} \cdots$].

"Queste due ultime equazioni manifestano una nuova, e bellissima proprietà del cerchio, ciascun arco del di cui quadrante à per suo elemento $\frac{dt}{1+t^2}$, quando la $t$ denota la tangente dell'arco medesimo.

"Scolio IV.—Se l'arco di cerchio fosse equale al quadrante, allora la $t$ diverrebbe infinita, e per avere il logaritmo equale al quadrante nulla gioverebbe l'equazione (8). In questo caso si divida per mezzo lo stesso quadrante, e la tangente dell'arco sudduplo di esso sarà eguale all'unità;

The relation $\pi/2 = -\sqrt{-1}\log\sqrt{-1}$ was known to Euler as early as 1728, since, on December 10 of that year he wrote as follows to Jean Bernoulli[1]: "Sit radius circuli $a$, sinus $y$, cosinus $x$, erit ex methodo tuâ quadraturam circuli ad logarithmos reducendi, area sectoris $= \dfrac{aa}{4\sqrt{-1}}\log\dfrac{x+y\sqrt{-1}}{x-y\sqrt{-1}}$, et posito $x = 0$, habebis quadrans circuli $\dfrac{aa}{4\sqrt{-1}}\log(-1)$." Hence, Euler demonstrated that $\dfrac{\pi a^2}{4} = \dfrac{a^2}{4\sqrt{-1}}\log(-1)$, from which one readily deduces

$$\frac{\pi}{2} = -\sqrt{-1}\log\sqrt{-1}.$$

In his reply, Jean Bernoulli called attention[1] to the identity

$$\int_0^{\pi/4}\frac{a^2dx}{2\sqrt{a^2-x^2}} = \frac{a^2}{4}\frac{\log\sqrt{-1}}{\sqrt{-1}},$$

and observed at the same time that the integral is equal to one eighth of a circle of radius $a$. It follows immediately that

$$\frac{\pi a^2}{8} = \frac{a^2}{4}\frac{\log\sqrt{-1}}{\sqrt{-1}}, \text{ or } \frac{\pi}{2} = -\sqrt{-1}\log\sqrt{-1};$$

but Jean Bernoulli did not himself draw this conclusion because he believed that $\log\sqrt{-1}$ was zero.[2]

pongasi poscia 1 in luogo di $t$ nel secondo membro dell'equazione (8), e si avrà il logaritmo eguale all'arco sudduplo del quadrante, e sarà log. $(A^2-B^2)$. Quindi si avrà lo stesso quadrante $-2\,l.(A^2-B^2) = \log.(A^2-B^2)^2$."

In some notes written by Count Fagnano's son, G. F. Fagnano (1715–1797) in 1761, the following formula occurs (*Opere*, tome 3, 1912, p. 30):

$$\text{Quadrante} = 2\,L.\left(\frac{1-1\sqrt{-1}}{1+1\sqrt{-1}}\right)^{\frac{1\sqrt{-1}}{2}}.$$

It is shown that this reduces to

$$\text{"Quadrante} = 2\,L.(-1\sqrt{-1})^{\frac{1\sqrt{-1}}{2}} = \sqrt{-1}\,L. -\sqrt{-1}$$

la qual'Espressione con cede in bellezza alla Bernulliana." On page 34 is the following corollary:

"E facile dedurre da tale dottrina, che quantunque L. $-$ 1 abbia un'infinità di valori tutti immaginarj; siccome nell'Equazione v.g.

$$\frac{\text{Circonf.}}{\text{Diam.}} = \frac{\text{L.} -1}{\sqrt{-1}}$$

uno solo è il valore della Circonferenza, uno solo è il valore del Diametro, e uno solo il valore della radice di $-$ 1, così uno solo degli infiniti valori di L $-$ 1 salva la suddetta equazione."

[1] G. Eneström, *Bibliotheca Mathematica*, 1899, p. 46.

[2] This is discussed at length by Euler in "De la controverse entre Mrs. Leibnitz & Bernoulli sur les Logarithmes des nombres négatifs et imaginaires" (presented to the Berlin Academy in 1747), *Mém. de l'acad. d. sc. de Berlin*, vol. 5 (1749), 1751, pp. 146–148. On page 147 we find: "Or M. Bernoulli ayant si hereusement réduit la quadrature du cercle aux logarithmes des nombres

The discovery by Euler in this connection that log $n$ has an infinite number of logarithms, which are all imaginary except when $n$ is a positive number, is a striking indication of his clear thinking and genius.[1] In the course of his paper "Recherches sur les racines imaginaires des équations"[2] we find (pages 272–276) a discussion of the problem: "Une quantité imaginaire étant élevée à une puissance dont l'exposant est aussi imaginaire, trouver la valeur imaginaire de cette puissance." Assuming

$$(a + b\sqrt{-1})^{m+n\sqrt{-1}} = x + y\sqrt{-i}$$

he finds

$$x = c^m e^{-2\lambda n\pi - n\phi} \cos(2\lambda m\pi + m\phi + n \log c),$$

$$y = c^m e^{-2\lambda n\pi - n\phi} \sin(2\lambda m\pi + m\phi + n \log c),$$

where $\lambda$ is a positive or negative integer, $c = \sqrt{(a^2 + b^2)}$ and $\cos(2\lambda\pi + \phi) = a/c$, $\sin(2\lambda\pi + \phi) = b/c$.[3] Euler's fourth corollary to this result is as follows:

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imaginaires, si le logarithme de $\sqrt{-1}$ etoit $= 0$, toute cette belle découverte seroit fausse; par laquelle il a fait voir, que le rayon est à la quatrieme partie de la circonference, comme $\sqrt{-1}$ à $\log\sqrt{-1}$. Donc posant le rapport du diametre à la circonference $= 1 : \pi$, il sera $\frac{1}{2}\pi = \dfrac{\log\sqrt{-1}}{\sqrt{-1}}$, & pourtant $\log\sqrt{-1} = \frac{1}{2}\pi\sqrt{-1}$, ce qui seroit absurde s'il étoit $\log\sqrt{-1} = 0$. Il n'est pas donc vray que $\log\sqrt{-1} = 0 \cdots$."

In a summary Euler stated (*l.c.*, p. 175):

| "Les valeurs de | seront celles-cy à l'infini | | | | | |
|---|---|---|---|---|---|---|
| $\dfrac{\log(+\sqrt{-1})}{\sqrt{-1}}$ | $+\frac{1}{2}\pi$; | $+\frac{5}{2}\pi$; | $+\frac{9}{2}\pi$; | $+\frac{13}{2}\pi$; | $+\frac{17}{2}\pi$; | &c. |
| | $-\frac{3}{2}\pi$; | $-\frac{7}{2}\pi$; | $-\frac{11}{2}\pi$; | $-\frac{15}{2}\pi$; | $-\frac{19}{2}\pi$; | &c. |
| $\dfrac{\log(-\sqrt{-1})}{\sqrt{-1}}$ | $+\frac{3}{2}\pi$; | $+\frac{7}{2}\pi$; | $+\frac{11}{2}\pi$; | $+\frac{15}{2}\pi$; | $+\frac{19}{2}\pi$; | &c. |
| | $-\frac{1}{2}\pi$; | $-\frac{5}{2}\pi$; | $-\frac{9}{2}\pi$; | $-\frac{13}{2}\pi$; | $-\frac{17}{2}\pi$; | &c." |

[1] A discussion of his work in this connection may be found in F. Cajori's "History of logarithms" in this MONTHLY, *1913*, 75–84.

[2] *Mém. de l'acad. d. sc. de Berlin*, vol. 5 (1749), 1751, pp. 222–288. The memoir seems to have been presented to the Academy in 1746 (G. Eneström, *Verzeichnis der Schriften Leonard Eulers*, Erste Lieferung, 1910, p. 43).

[3] The proof that $(a + b\sqrt{-1})^{m+n\sqrt{-1}}$ is expressible in the form $x + y\sqrt{-1}$ is often referred to as "d'Alembert's theorem" [*e.g.*, *Annales de Mathématiques Pure et Appliquées* (Gergonne), July, 1913, p. 20]. He discussed the question in his *Reflexions sur la cause générale des vents*, Paris, 1747, p. 142, and in *Mém. de l'acad. de sc. de Berlin*, vol. 2 (1746), 1748, p. 192; cf. *Opuscules Mathématiques* . . . par M. d'Alembert, Paris, tome 1, 1761, p. 225 and tome 5, 1768, pp. 213–214. Part of this discussion was developed in L. A. de Bougainville, *Traité du calcul intégral, pour servir de suite à l'analyse des infiniment petits de l'Hôpital*, tome 1, Paris, 1754, pp. 42f. With Bernoulli, D'Alembert claimed that $\log\sqrt{-1} = 0$.

The "theorem" was also discussed by J. B. Labey in his notes to Euler, *Introduction à l'Analyse Infinitésimale*, tome 1, Paris, 1796, pp. 326–327; by Lagrange in his *Traité de la Résolution des Equations Numériques*, nouvelle édition, Paris, 1808, note IX; and by du Bourguet, and Gergonne, in *Annales de Mathématiques Pures et Appliquées*, tome 4, 1813, pp. 20–25. In *The Ladies' Diary*, 1833, p. 48, problem 1567 was, in effect: "When is $(a + b\sqrt{-1})^{m+n\sqrt{-1}}$ a real quantity? Determine whether all functions of $a + b\sqrt{-1}$ can be reduced to $A + B\sqrt{-1}$, or not." This was answered in the *Diary* for 1834, pp. 44–45. See also A. Cayley, *Proc. London Math. Soc.*, vol. 2, p. 54; *Coll. Math. Papers*, vol. 6, p. 68.

"Si $a = 0$; $m = 0$, & $b = 1$, il sera $c = 1$ & $\phi = \frac{1}{2}\pi$ d'où l'on tirera cette transformation:

$$(\sqrt{-1})^{n\sqrt{-1}} = e^{-2\lambda n\pi - \frac{1}{2}n\pi}$$

ou bien

$$(\sqrt{-1})^{\sqrt{-1}} = e^{-2\lambda\pi - \frac{1}{2}\pi},$$

qui est d'autant plus remarquable, qu'elle est réelle, & qu'elle renferme même une infinité de valeurs réelles differentes. Car posant $\lambda = 0$, on aura en nombres

$$(\sqrt{-1})^{\sqrt{-1}} = 0{,}2078795763507."$$

Euler gives a similar value in the last paragraph of a letter to Goldbach, dated June 14, 1746:[1] "Letztens habe gefunden, dass diese expressio $\sqrt{-1}^{\sqrt{-1}}$ einen valoren realem habe, welcher in fractionibus decimalibus = 0,2078795763, welches mir merkwurdig zu seyn scheinet."

But Euler's results were not generally known and accepted. For example, more than sixty-five years later we find Argand, notable for his geometrical interpretation of imaginary quantities,[2] stating that $\sqrt{-1}^{\sqrt{-1}}$ offers a simple example of a quantity which is irreducible to the form $x + y\sqrt{-1}$.[3] "He did not," as Hamilton remarks,[4] "anticipate De Morgan's theory of logometers."[5]

Among manuscripts published after Gauss's death (1855) were certain ones dealing with lemniscate functions. In one of these, values of $e^{-\pi}$, $e^{-(\pi/4)}$, $e^{-(9\pi/4)}$, $e^{\pi/2}$, and $11 \cdot e^{-(\pi/2)}$ are computed in an interesting manner (*Carl Friedrich Gauss Werke*, vol. 3, 1864, pp. 426–432). The values are as follows:

$$e^{-\pi} = 0.0432139182\ 6377224977\ 4417737171\ 7280112757\ 2810981063,$$

$$e^{-(\pi/4)} = 0.4559381277\ 6599623676\ 5921294728\ 0294194166\ 0436523820,$$

$$e^{-(9\pi/4)} = 0.0008514383\ 4280515803\ 5852453295\ 4846487994\ 1872486024\ 8176915,$$

$$e^{\pi/2} = 4.8104773809\ 6535165547\ 3044648993\ 1536, \text{ and}$$

$$11 \cdot e^{-(\pi/2)} = 2.2866753398\ 58378 \text{ which gives}$$

$e^{-(\pi/2)} = .2078795763\ 50762$. On pages 418–419 (*l.c.*) Gauss sets down values for $2e^{-(\pi/4)}$ to 39 places of decimals, $2e^{-(9\pi/4)}$ to 27 places, $2e^{-\pi}$ to 40 places, $2e^{-(25\pi/4)}$ to 32 places, $2e^{-(49\pi/4)}$ to 28 places, $2e^{-4\pi}$ to 35 places, and $2e^{-9\pi}$, $2e^{-16\pi}$ to 27 places. The last five of these values are as follows:

[1] *Correspondance Mathématique et Physique de quelques célèbres géometres du XVIIIème siècle* . . . publiée . . . par P. H. Fuss. Tome 1, St. Pétersbourg, 1843, p. 383.

[2] *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques*, Pairs, 1806.

[3] *Annales de Mathématiques Pures et Appliquées*, tome 4, 1813, p. 146. Servois called attention to Euler's result (*l. c.* 1814, p. 231) in correcting this error. Details of the discussion in this connection are given in the volume of A. S. Hardy's English translation of Argand's *Essai* (Van Nostrand Science Series), New York, 1881.

[4] W. R. Hamilton, *Lectures on Quaternions*, Dublin, 1853, p. (56).

[5] See A. De Morgan, *Trigonometry and Double Algebra*, London, 1849, pp. 129–137; also R. B. Hayward, *The Algebra of Coplanar Vectors and Trigonometry*, London, 1892, pp. 119f.

$$2e^{-(25\pi/4)} = 0.0000000059\ 3851399312\ 9644497731\ 18$$

$$2e^{-(49\pi/4)} = 0.0000000000\ 0000003867\ 40505991$$

$$2e^{-4\pi} = 0.0000069746\ 8471241799\ 0983550387\ 96535$$

$$2e^{-9\pi} = 0.0000000000\ 0105109703\ 5201288$$

$$2e^{-16\pi} = 0.0000000000\ 0000000000\ 0295807$$

It will be observed that the values for $e^{-\pi}$ and $e^{-(\pi/2)}$ agree exactly with those given above by Professor Uhler, but that the values for $e^{\pi/2}$ differ—from the twenty-third decimal place on. Bastien's value, given below, appears to indicate that the value in Professor Uhler's paper is the correct one to twenty-eight places of decimals at least. By squaring Gauss's value for $e^{-(\pi/4)}$ Professor Uhler found the result to check to forty-seven places of decimals with his own value for $e^{-(\pi/2)}$.

Schellbach showed[1] in 1832 that many convergent series for $\pi$ could be derived from such relations as

$$\pi = \frac{2}{i}\log i = \frac{2}{i}\log\frac{1+i}{1-i} = \frac{2}{i}\log\frac{(2+i)(3+i)}{(2-i)(3-i)} = \frac{2}{i}\log\frac{(5+i)^4(-239+i)}{(5-i)^4(-239-i)}$$

$$= \frac{2}{i}\log\frac{(10+i)^3(-515+i)^4(-239+i)}{(10-i)^3(-515-i)^4(-239-i)}.$$

Benjamin Peirce referred,[2] in 1882, to "the mysterious formula"

$$i^{-i} = e^{\pi/2} = 4.810477381.$$

The phrase "l'équation symbolique et mystérieuse

$$\frac{\pi}{2}\sqrt{-1} = \text{Log.}\,(\sqrt{-1})\text{"}$$

was employed by J. F. Francais[3] in September, 1813.

In the introduction to *Table d'Interpolation pour le calcul des parties proportionnelles faisant suite* aux *Tables de Logarithmes* . . . par L. Schrön precédé d'une introduction française par J. Houel (Paris, 1891) the following number is evaluated

$$e^{-\pi} = 0.043213918263772248 \cdots.$$

In 1919, Brocard suggested[4] that "la connaisance des nombres $e^\pi$ and $\pi^e$ donnerait peut-être l'indice de quelque relation entre $e$ et $\pi$." E. Chanzy found such values[5] to be $e^\pi = 23.1406926327787 \cdots$, and $\pi^e = 22.4591577183 \cdots$;

[1] *Journal für die reine und angewandte Mathematik*, vol. 9, pp. 404–405.
[2] B. Peirce, *Linear Associative Algebra*, New York, 1882, p. 5.
[3] *Annales de Mathématiques Pures et Appliquées*, vol. 4, p. 67.
[4] *L'Intermédiaire des Mathématiciens*, vol. 26, p. 73, question 4935.
[5] *Sphinx-Oedipe*, August, 1920, pp. 127–128.

and L. Bastien, with the aid of his tables of logarithms to 32 places of decimals found[1]

$$e^{\pi} = 23.140692632779269005729086367 9,$$
$$e^{\pi/2} = 4.8104773809653516554730356667,$$
$$\pi^{e} = 22.459157718361045473427152204 5.$$

The symbol $i$ for $\sqrt{-1}$ was first used by Euler in a "M.S. Academiæ exhibit. die 5 Maii 1777"[2] printed posthumously in 1794.[3] This notation was adopted by Gauss in 1801.[4]

The symbol $\pi$ for the ratio of the circumference of a circle to its diameter was first used by W. Jones in 1706.[5] It was probably suggested to Jones by Oughtred who employed the symbol in a different sense.[6] Euler's first use of the symbol was in 1737[7]; up to that time he had used the letter $p$.

The symbol $e$ for the number, defined by the series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \cdots$, was first used by Euler in 1731.[8]

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# AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

## 2. Dupin as Secretary of the Ionian Academy.

Among the most interesting but by no means most scholarly of the French mathematicians of the first part of the nineteenth century was that mélange of economist, politician, geometrician, and popularizer of science, Pierre-Charles-François Dupin (1784–1873). To the mathematician he is chiefly known as the favorite pupil of Monge and as his biographer; for his *Développement de géométrie*

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[1] *L'Intermediaire des Mathematiciens*, vol. 27, p. 65, May–June (not published till October), 1920. In the value for $\pi^e$ given above the number "1" has been inserted in the seventh place from the end where "$r$" appears in the original.

[2] L. Euler, "De formulis differentialibus angularibus maxime irrationalibus, quas tamen per logarithmos et arcus circulares integrare licet."

[3] L. Euler, *Institutiones calculi integralis*, vol. 4, 1794, pp. 183–184. The symbol $i$ is introduced on p. 184.

[4] C. F. Gauss, *Disquisitiones arithmeticae*, 1801, p. 337; *Werke*, vol. 1, 1870, p. 414.

[5] W. Jones, *Synopsis Palmiorum Matheseos*, London, 1706, p. 263.

[6] William Oughtred (1574–1660) in his *Clavis Mathematica* of 1647, etc., and in his *Theorematum in libris Archimedes de Sphaera et Cylindro Declaratio*, Oxford, 1652, frequently employs the symbol $\delta : \pi$ or $\pi : \delta$ (in modern notation) for the ratio of the semi-diameter to the semi-periphery or of semi-periphery to semi-diameter. It is noticeable that these letters are *never* used separately, that is, $\pi$ is *not* used for "Semiperipheria," as Tropfke suggests (*Geschichte der Elementar-Mathematik*, vol. 2, 1903, p. 135). Oughtred states specifically in his "Theorematum": "$\frac{\pi}{\delta} R$, est semiperipheria circuli cujus Radius est $R$." In 1697 David Gregory used (*Philosophical Transactions*, vol. 19, p. 652) $\pi/\rho$ to designate the ratio of the circumference to the diameter.

[7] L. Euler, "Variae observationes circa series infinitas," *Comment. acad. sc. Petrop.*, vol. 9 (1737), 1744, p. 165.

[8] *Corresp. math. et phys* . . . par Fuss, vol. 1, 1843, p. 58: "e denotat hic numerum, cujus logarithmus hyperbolicus est = 1."